General Formulation for Quantitative G-factor Calculation in GRAPPA Reconstructions

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Introduction: G-factor

- Multi-coil arrays
  - Higher SNR, shorter scan time (pMRI)
  - non-uniform SNR

- SNR decrease after parallel imaging recon

\[ SNR^{\text{pMRI}} = \frac{SNR^{\text{full}}}{g\sqrt{R}} \]

  - R: reduction factor
  - g: geometry (g)–factor
    - Reflects spatially variant noise enhancement
    - Depends on R, encoding capability of receiver array

- Practical analytical g-factor description
  - Exist for SENSE, SMASH, PARS,
  - Doesn’t exist for GRAPPA
Introduction: Estimation of G-factor

- Gold standard
  - Fully encoded images with identical parameters

    \[ SNR_{full\ pMRI} = \frac{\text{mean}_{full\ pMRI}}{\text{std}_{full\ pMRI}} \]

    \[ g_\rho = \frac{SNR^{full}_{\rho}}{SNR^{pMRI}_\rho \sqrt{R}} \], \( \rho \): pixel index

- Pseudomultiple replica SNR measurement
  - 1 fully encoded image, additional noise-only dataset

- \( N \geq 100 \) images/recons, time consuming

Introduction: Outline of this paper

- General framework for quantitative g-factor in GRAPPA
  - Use GRAPPA recon weights in image space
  - Single coil GRAPPA images & combined images (e.g. SOS)
  - Essential to consider noise correlations

- Accuracy and general applicability demonstrated
Study Group Discussion

- Introduction
  - Enough background?
  - Brief summary of whole paper 😊

Theory: GRAPPA Review

- Recon k-space for each coil, then combine
- Weights determination

\[ w \cdot S_{src} = S_{trg} \quad \Rightarrow \quad w = S_{trg} \cdot p\text{inv}(S_{src}) \]
Theory: GRAPPA Review

- GRAPPA recon weights in image space

\[ \sum_{l=1}^{Nc} W_{kl} \ast S_l^{\text{red}} = I_k^{\text{acc}} \]

\[ I_k^{\text{acc}} = \sum_{l=1}^{Nc} W_{kl} \cdot I_l^{\text{red}} \]

FIG. 2.

Convolution in k-space

Multiplication in image space

Theory: Noise Propagation in GRAPPA

Separate the noise term from the signal

\[ I_k^{\text{acc}} + n_k^{\text{acc}} = \sum_{l=1}^{Nc} W_{kl} \cdot (I_l^{\text{red}} + n_l^{\text{red}}) \]

\[ n_k^{\text{acc}} = \sum_{l=1}^{Nc} W_{kl} \cdot n_l^{\text{red}} \]

\[ \sigma^2(n_k^{\text{acc}}) = \sigma^2(\sum_{l=1}^{Nc} W_{kl} \cdot n_l^{\text{red}}) \]

\[ = \sum_{l=1}^{Nc} |W_{kl}|^2 \cdot \sigma^2(n_l^{\text{red}}) + 2 \sum_{l=1}^{Nc} \sum_{m=l+1}^{Nc} |W_{kl} W_{lm}| \cdot \sigma^2(n_l^{\text{red}}, n_m^{\text{red}}) \]

\[ = |W \cdot \Sigma^2 \cdot W^H|_{dd} \]

- \( I_k^{\text{acc}} \): noise-free accelerated GRAPPA image/folded image in the k-th coil.
- \( n_k^{\text{acc}} \): noise in the accelerated GRAPPA image/folded image in the k-th coil.
- \( N_c \): # of coils.
- \( \sigma^2(n_l^{\text{red}}) \): noise variance in channel \( l \).
- \( \sigma^2(n_l^{\text{red}}, n_m^{\text{red}}) \): noise covariance between channel \( l \) and \( m \).
- \( \Sigma^2 \): noise covariance matrix
- \( W_{kl} \): GRAPPA weights in image space.
- \( W \): GRAPPA weights matrix in image space
- \( H \): transpose complex conjugate
Theory: Noise Propagation in GRAPPA

- Variance of accelerated GRAPPA k-th coil image
  \[ \sigma^2(n_k^{acc}) = |W \cdot \Sigma^2 \cdot W^H| \]

- Variance of fully encoded k-th coil image
  \[ \sigma^2(n_k^{full}) = \frac{1}{R} | \Sigma^2 | \]

- G-factor for the k-th coil image
  \[ g_k = \frac{\text{SNR}_{full}}{\text{SNR}_{acc}} \sqrt{\frac{R}{\sigma(n_k^{acc})}} = \sqrt{\frac{W \cdot \Sigma^2 \cdot W^H}{\Sigma^2}} \]

\[ W = \begin{pmatrix} W_{11} & \ldots & W_{1N_c} \\ \vdots & \ddots & \vdots \\ W_{N_c,1} & \ldots & W_{N_c,N_c} \end{pmatrix}, \quad \Sigma^2 = \begin{pmatrix} \sigma_{11} & \ldots & \sigma_{1N_c} \\ \vdots & \ddots & \vdots \\ \sigma_{N_c,1} & \ldots & \sigma_{N_c,N_c} \end{pmatrix}, \quad p = (p_1, \ldots, p_{N_c})^T, I = \text{eye}(N_c) \]

Theory: Noise Propagation in GRAPPA

- G-factor for the k-th coil image
  \[ I_{k\text{acc}} = \sum_{l=1}^{N_c} W_{kl} \cdot I_{l\text{red}} \implies g_k = \sqrt{\frac{|W \cdot \Sigma^2 \cdot W^H|}{\Sigma^2}} \]

- G-factor for the combined image
  \[ I_{\text{comb}} = \sum_{k=1}^{N_c} p_k \cdot I_{k\text{acc}} = \sum_{k=1}^{N_c} p_k \cdot \sum_{l=1}^{N_c} W_{kl} \cdot I_{l\text{red}} \implies g_{\text{comb}} = \sqrt{\frac{|(p^T \cdot W) \cdot \Sigma^2 \cdot (p^T \cdot W)^H|}{|(p^T \cdot I) \cdot \Sigma^2 \cdot (p^T \cdot I)^H|}} \]

- SOS recon
  - \( p_k \) (in vector \( p \)) = \( l_k' / l_{\text{GRAPPA}} \)
  - \( l_k' \) can be \( l_k^{\text{acc}} \) or determined from ACS data; : conjugate

- SOS GRAPPA g-factor
  - accurate estimate for basically all commonly used combining methods
Theory: Noise Propagation in GRAPPA

- Multiple kernels or ACS data included in final recon
  - For each kernel, noise propagates according to \( \sigma^2(n_{\text{new}}) = |W \cdot \Sigma^2 \cdot W^\dagger|_{\text{acq}} \)
  - Determine g-factor for each kernel, then weight by \( f_m \) and \( R_m \)

\[
\sigma_{\text{eff}} = \sqrt{\sum_{m=1}^{N_k} f_m \cdot R_m \cdot g_m^2}
\]

\[\text{with} \quad R_{\text{eff}} = \left( \sum_{m=1}^{N_k} f_m R_m \right)^{-1}\]

- \( N_k \): # of kernels used
- \( f_m \): fraction of k-space the m-th kernel has been applied to
- \( R_m \): reduction factor for the m-th kernel
- \( g_m \): the g-factor (uncombined or combined) for the m-th kernel
- For ACS data:
  - \( f_1 \) = #of ACS lines / #of lines in fully encoded k-space
  - \( R_1 = 1 \)
  - \( g_1 = 1 \)

Study Group Discussion

- Theory
  - Brief review of GRAPPA
  - Equation deduction
  - Notation
Materials and Methods

- **Scan**
  - **General setup**
    - 1.5T Siemens
    - 12-channel head coil
    - offline Recon & calculations
    - 1 fully encoded dataset
    - An additional noise-only image
  - **Phantom experiments (2D)**
    - 1 fully encoded 2D image: 256 x 256
    - Mimicked GRAPPA acquisitions with R=2,3,4
    - 4 x 5(PE x FE) GRAPPA kernel, 32 x 32 ACS
  - **In Vivo Experiments (3D)**
    - 1 fully encoded 3D axial MPRAGE(magnetization prepared rapid gradient echo) head dataset
    - Mimicked rectangular(R=2 x 2) and CAIPIRINA-type(R=2 x 2) acquisitions
    - 3 x 3 x 3 3D GRAPPA kernel, 24 x 24 x 32 ACS block

- **Simulation (non-Cartesian)**
  - Simulated 8-element one-ring head coil
  - Simulated PROPELLER dataset with normal complex noise
  - Mimicked accelerated PROPELLER acquisitions with R=2,3,4
  - GRAPPA recon: each blade reconed separately, finally gridded to Cartesian

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Materials and Methods

- **Gold standard method for g-factor (pseudomultiple-replica method)**

\[
\text{SNR}_{\text{GRAPPA}}^\text{full} = \frac{\text{mean}_{\text{full}}}{\text{std}_{\text{full}}} \quad g_{\rho} = \frac{\text{SNR}_{\rho}^\text{full}}{\text{SNR}_{\rho}^\text{GRAPPA} \sqrt{R}}, \quad \rho: \text{pixel index}
\]

- For both uncombined and combined images
Materials and Methods

G-factor Estimation from GRAPPA Weights

- GRAPPA weights: from ACS data in k-space
- Noise correlations: from additional noise scan
- For the k-th coil image

\[ g_k = \frac{\sqrt{|W \cdot \Sigma^2 \cdot W^H|_k}}{\sqrt{|\Sigma^2|_k}} \]

- For SOS combined or B1 normalized combined images
  \[ p_k = \frac{l_k}{l_{SOS}} \] from ACS or reconed GRAPPA images

\[ g_{\text{comb}} = \frac{\sqrt{|(p^* \cdot W) \cdot \Sigma^2 \cdot (p^* \cdot W)^H|}}{\sqrt{|(p^* \cdot \mathbf{I}) \cdot \Sigma^2 \cdot (p^* \cdot \mathbf{I})^H|}} \]

Study Group Discussion

- Materials and Methods
Results

1. Single coil images
2. Combined image
3. @ various reduction factors
4. GRAPPA vs. SENSE
5. In vivo 3D(2D accelerated)
6. VD-GRAPPA (multikernel GRAPPA)
7. Non-Cartesian GRAPPA

phantom          In vivo          simulation

Results

1. Single coil images

FIG. 3. Quantitative g-factor maps for uncombined GRAPPA images (R = 3, 12 channel head array) derived from (a) pseudomultiple replica images series derived from a phantom and extra noise scan and (b) directly from the GRAPPA reconstruction weights.
Results

2. Combined image

FIG. 4. Quantitative $R = 4$ SOS-combined GRAPPA g-factor maps derived from (a) pseudomultiple replica image series and (b) directly from the GRAPPA reconstruction weights including noise correlations and (c) neglecting noise correlations.

Results

3. @ different $R$

FIG. 5. Quantitative GRAPPA g-factor maps at various reduction factors $R = 2, 3, 4$ (top row) and corresponding SOS-combined GRAPPA reconstructions (bottom row).
Results

4. GRAPPA vs. SENSE

FIG. 6. Comparison between SENSE and GRAPPA reconstruction performance at a reduction factor $R = 3$. In (a) the SENSE reconstruction and corresponding g-factor map is displayed. In (b) the GRAPPA reconstruction and corresponding SOS-GRAPPA g-factor is displayed.

Results

5. In vivo 3D(2D accelerated)

FIG. 7. Central partition in the sagittal orientation of a 4-fold accelerated 3D MPRAGE experiment after (a) rectangular ($R = 2 \times 2$) and (b) CAIPRINHA-type ($R = 2 \times 2(1)$) acquisition with sampling positions shifted with respect to each other. Displayed are the SOS-combined GRAPPA reconstructions, the corresponding GRAPPA g-factor maps, and 2D phase encoding schemes.
Results

6. VD-GRAPPA (multikernel GRAPPA)

- Each blade reconstructed separately
- G-factor map derived for each blade
- Single blade g-factors rotated by rotation angle in k-space
- G-factor maps combined by Eq.[13]

**FIG. 8.** Variable density acquisition with an effective reduction factor of R = 2.18. a: VD-GRAPPA reconstruction. b: VD acquisition scheme. c: G-factor map calculated from pseudomultiple replica series. d: GRAPPA g-factor map derived directly from the GRAPPA reconstruction weight sets (R = 1, 2, 3, 4).

Results

7. non-Cartesian GRAPPA

**FIG. 9.** Reconstruction results from b a) an accelerated (R = 4) simulated PROPELLER acquisition and a) the fully sampled reference PROPELLER image (8 blades, 32 phase encoding lines per blade). c: G-factor map calculated from pseudomultiple replica series. d: GRAPPA g-factor map derived by calculating the g-factor for each blade separately according to Eq. [12] followed by a weighted combination of the single blade g-factors according to Eq. [13].
Study Group Discussion

- Results
  - Figure captions.
  - Convincing results?
  - Sufficient? Too many?
  - Order of results?

Current Order of Results

1. Single coil images
2. Combined image
3. @ various reduction factors
4. GRAPPA vs. SENSE
5. In vivo 3D(2D accelerated)
6. VD-GRAPPA (multikernel GRAPPA)
7. Non-Cartesian GRAPPA
### Possible New Order of Results

1. Single coil images
2. Combined image @ various reduction factors
3. VD-GRAPPA (multikernel GRAPPA)
4. Compare GRAPPA with other pMRI methods
5. In vivo 3D(2D accelerated)
6. Non-Cartesian GRAPPA

![Possible New Order of Results](image)

### Discussion

1. **Summary of methods & results**
   - Practical, fast, robust quantification of non-uniform noise enhancement in GRAPPA recon for both uncombined and combined images
   - Excellent agreement with gold standard method

2. **Advantage of proposed method**
   - No coil sensitivity required
   - Quantitatively estimate noise enhancement PRIOR to recon

3. **Things need to be careful with**
   - Account for noise correlations between coils
## Discussion

### 4. (Potential) applications

- **Kernel optimization**
- Compare performance of different acquisition strategies (e.g. rectangular vs. 2D CAIPIRINHA)
- Compare different pMRI methods (GRAPPA, SENSE, SMASH)
- Multi-kernel GRAPPA
  - Noise in source points uncorrelated for each kernel application
  - Could calculate contribution of each kernel g-factor
- Non-Cartesian GRAPPA
  - PROPELLER
  - Radial, spiral
- Generalized pseudo-Cartesian GRAPPA
  - For arbitrary trajectories
  - Shift non-Cartesian data to nearest Cartesian locations, followed by Cartesian multiple kernel GRAPPA
  - Perfect candidate for generalized g-factor estimation

<table>
<thead>
<tr>
<th>Shown in this paper</th>
<th>Not shown/Future work</th>
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## Conclusion

Brief summary of the whole paper
Study Group Discussion

- Discussion
  - Enough?
  - Computation time comparison with gold standard method?
  - How to get accurate noise correlations?

- Conclusion

Thank you!

Questions?